Journal of Engineering and Applied Science

Contents are available at www.jeas.ruet.ac.bd

The Lindstedt-Poincare Method for Solving a Nonlinear Pendulum like Oscillator

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ARTICLE INFORMATION

Received date: 12 Mar 2026
Revised date: 09 May 2026
Accepted date: 07 Jun 2026

Keywords

Lindstedt-Poincare method (LP)
Pendulum like oscillator
Nonlinear oscillator
Periodic Solution.

ABSTRACT

The periodically addressed steady-state response of a nonlinear pendulum like oscillator is investigated in this study using the classical Lindstedt–Poincare technique. The oscillator is obtained by a systematic transformation of the nonlinear differential equation $\ddot{u}+(1+\dot{u}^2)u=0$, which cannot be easily handled in its original form because of its high nonlinearity. The system can be simplified to a corresponding pendulum like nonlinear oscillator of the form $\ddot{x}+\tan x=0$, along with consistently transformed initial conditions. This change does not alter any of the basic dynamical characteristics, but drastically simplifies the analytical treatment. Nonlinear oscillatory systems are common in physics and engineering, and the equations governing their behavior do not often have analytical solutions. Many traditional perturbation techniques can give secular terms that are increasing with time, and provide non-uniform and physically unrealistic approximations. In order to solve this problem, the Lindstedt-Poincare method is used to remove the secular terms by adding a perturbation expansion to the system frequency, and this ensures the uniformly valid periodic solutions. The Lindstedt-Poincare method is used to compute approximate analytical expressions of the solution and frequency of the transformed oscillator. The accuracy and effectiveness of the proposed approach are demonstrated through comparisons with numerical solutions obtained by the fourth-order Runge–Kutta method, as well as with results from the harmonic balance method, frequency–amplitude formulation, and the parameter expansion method. The comparisons show excellent agreement, confirming the reliability and robustness of the present analysis for nonlinear oscillatory systems.

1. Introduction

Nonlinear differential equations are used as a fundamental tool to model complex phenomena in various fields including physics, chemistry, biology, economics and engineering sciences. Nonlinearity is an inherent property of many real-world systems because of geometric configurations, material behavior, large deformations or external influence. Consequently, nonlinear models cannot be ignored when characterizing

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a broad spectrum of processes such as oscillatory movement, propagation of waves, population dynamics and chemical interactions. Nonlinear differential equations, however, unlike linear systems, rarely have closed-form solutions and their behavior as solutions depends highly on system parameters, leading to complex system behavior such as frequency changes dependent on amplitude and qualitative changes in motion.

The challenging nature of determining precise solutions has spurred much research on approximate methods of analysis. These techniques are intended to give correct descriptions of system behavior without losing physical insight, which is often not available in purely numerical simulations. Classical perturbation methods [1] are among the oldest and most used methods, and they are based on the fact that there is a small parameter, and solutions are constructed in terms of small parameters, as asymptotic series. Although such methods are successful in weakly nonlinear regimes, they are often plagued by the appearance of secular terms time-dependent components that grow without bound and limiting the validity of the approximation to short time intervals.

To overcome this weakness, there are modified perturbation techniques, one of which is the Lindstedt-Poincare method [1-9]. This approach is effective in quadrupling the secular terms and obtaining uniformly valid periodic solutions. Due to this ability, it has become a standard analysis tool to investigate nonlinear oscillators, especially those with an amplitude-dependent frequency behavior.

Along with the methods of perturbation, a number of alternative approaches have been suggested to address nonlinear systems with different levels of complexity. The harmonic balance method (HBM) [10-13] is an approximation method of periodic responses based on truncated Fourier expansions, which reduce the governing equations to solvable algebraic equations. Iterative schemes [14-16] offer a systematic refinement of approximate solutions, whereas the homotopy perturbation method (HPM) [17-18] combines the ideas of perturbation with those of homotopy theory to simplify highly nonlinear problems. The energy balance approach (EBM) [19-20], however, provides a more intuitive approach to energy, based on energy considerations in a cycle of motion. Moreover, the direct method of establishing relationships between the frequency and amplitude of oscillation, with no small parameters, the frequency-amplitude formulation method [21] has attracted attention, especially to strongly nonlinear oscillators. Correspondingly, the parameter expansion method [22] introduces an artificial parameter to re-formulate the problem so that the construction of rapidly convergent approximate solutions can be constructed even in the absence of naturally small quantities. The usefulness and appropriateness of these approaches is determined by the particular properties of the system, such as the nature and the intensity of nonlinearity.

Motivated by the need for an accurate analytical treatment of strongly nonlinear oscillatory systems, the present work focuses on a nonlinear model that can be recast into a pendulum type formulation through an appropriate transformation. This reformulation facilitates the application of analytical techniques while preserving the essential dynamical characteristics of the original system. The subsequent analysis employs the Lindstedt–Poincare method to construct approximate periodic solutions and to capture the associated frequency behavior. The validity of the obtained results is assessed through comparison with numerical computations and other established approximation methods, demonstrating the effectiveness of the approach in handling nonlinear oscillatory dynamics.

2. The Lindstedt-Poincare method [1]

Let us consider a nonlinear differential equation

$$\ddot{x} + \omega_0^2 x + \varepsilon f(x) = 0,\ x(0) = a, \dot{x}(0) = 0, \qquad (2.1)$$

where the over-dot denotes differentiation with respect to the time variable t, ω_0 represents the unperturbed frequency of the system, $f(x)$ is a nonlinear function of the displacement x and ε is a small perturbation parameter.

By a transformation of time $\tau = \omega t$, Eq. (2.1) is rewritten as

$$\omega^2 x'' + \omega_0^2 x + \varepsilon f(x) = 0,\ x(0) = a, x'(0) = 0, \quad (2.2)$$

where prime denotes differentiation with respect to transformed variable τ.

In the Lindstedt-Poincare method, the variable x and frequency ω^2 are assumed to be expanded in a power series of the small parameter ε as follows:

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots, \qquad (2.3)$$

and

$$\omega^2 = \omega_0^2 + \varepsilon \omega_1 + \varepsilon^2 \omega_2 + \cdots. \qquad (2.4)$$

Earlier it was chosen as $\omega = \omega_0^2 + \varepsilon \omega_1 + \varepsilon^2 \omega_2 + \cdots$. But Veronis [23], Burtons [24], Burton and Rahman [25] used the series Eq. (2.4) as it is superior to earlier equation.

Substituting the solution Eq. (2.3) and the frequency Eq. (2.4) into Eq. (2.2) and equating the coefficients of the same order of ε from both sides, the following linear equations for $x_0, x_1, x_2, \ldots$ are obtained as follows:

$$\omega_0^2 x_0'' + \omega_0^2 x_0 = 0, \qquad (2.5)$$

$$\omega_0^2 x_1'' + \omega_0^2 x_1 = -2\omega_0\omega_1 x_0'' - f(x_0), \qquad (2.6)$$

$$\omega_0^2 x_2'' + \omega_0^2 x_2 = -(2\omega_0\omega_2 + \omega_0^2) x_0'' - 2\omega_0\omega_2 x_1'' - x_1 \frac{\partial f(x_0)}{\partial x}. \qquad (2.7)$$

The previous initial conditions $x(0) = a, \dot{x}(0) = 0$ are replaced by

$$x_0(0) = a,\ x_i(0) = 0, for\ i = 1,\ 2, \cdots, \qquad (2.8)$$

$$x_0'(0) = 0,\ \ x_i'(0) = 0, for\ i = 1,\ 2, \cdots. \qquad (2.9)$$

Clearly, Eq. (2.5) - (2.7) are linear in nature and $x_0, x_1, \omega_1, x_2, \omega_2$ can be solved sequentially by the elementary techniques.

3. EXAMPLE

Let us consider the nonlinear differential equation [21, 22, 26]

$$\ddot{u} + (1 + \dot{u}^2)u = 0, u(0) = 0, \dot{u}(0) = b, \quad (3.1)$$

where over dots denote the differentiation with respect to t and for this oscillator, the unperturbed frequency is $\omega_0 = 1$.

Eq. (3.1) may be rewritten as

$$\frac{\ddot{u}}{(1+\dot{u}^2)} + u = 0, \quad (3.2)$$

or equivalently

$$\frac{d}{dt}[\tan^{-1}(\dot{u})] + u = 0. \quad (3.3)$$

Introducing the transformation

$$\dot{u} = \tan x, \quad (3.4)$$

Eq. (3.3) reduces to

$$\dot{x} + u = 0. \quad (3.5)$$

Differentiating Eq. (3.5) with respect to t, we obtain

$$\ddot{x} + \dot{u} = 0. \quad (3.6)$$

Substituting Eq. (3.4) into Eq. (3.6) yields

$$\ddot{x} + \tan x = 0. \quad (3.7)$$

The initial conditions are transformed as follows. At $t = 0$, Eq. (3.4) gives

$$\dot{u}(0) = \tan x(0) = b, \quad (3.8)$$

which implies

$$x(0) = \tan^{-1} b = a. \quad (3.9)$$

Moreover, from Eq. (14),

$$\dot{x}(0) = -u(0) = 0. \quad (3.10)$$

Thus, Eq. (3.1) is reduced to the equivalent initial value problem

$$\ddot{x} + \tan x = 0, x(0) = a, \ \dot{x}(0) = 0, \quad (3.11)$$

Where $a = \tan^{-1} b$. Including an artificial parameter $\varepsilon(= 1)$, the Eq. (20) can be rewritten as

$$\ddot{x} + \frac{\tan\sqrt{\varepsilon}x}{\sqrt{\varepsilon}} = 0, x(0) = a, \dot{x}(0) = 0. \quad (3.12)$$

Also, by a variable transformation, $\tau = \omega t$, Eq. (3.12) can be rewritten as follows

$$\omega^2 x'' + \frac{\tan\sqrt{\varepsilon}x}{\sqrt{\varepsilon}} = 0, x(0) = a, x'(0) = 0, \quad (3.13)$$

where prime denotes differentiation with respect to τ.

As reported by the Lindstedt-Poincare method, substituting Eq. (2.3) and Eq. (2.4) into Eq. (3.13) and equating the coefficients of $\varepsilon^0, \varepsilon^1, \varepsilon^2, ...$ the resulting linear differential equations for $x_0, x_1, x_2, ...$ are obtained as follows:

$$x_0'' + x_0 = 0, x_0(0) = a, x_0'(0) = 0, \quad (3.14)$$

$$x_1'' + x_1 = -\frac{x_0^3}{3} - x_0''\omega_1, \ x_1(0) = 0, x_1'(0) = 0, \quad (3.15)$$

$$x_2'' + x_2 = -\frac{2x_0^5}{15} - x_0^2 x_1 - x_1''\omega_1 - x_0''\omega_2, x_2(0) = 0, x_2'(0) = 0. \quad (3.16)$$

The approximated solution of Eq. (3.14) can be chosen as

$$x_0 = a \cos \tau. \quad (3.17)$$

Substituting the value of x_0 from Eq. (3.17) into Eq. (3.15) and simplifying, it becomes

$$x_1'' + x_1 = \frac{1}{12}(12a\omega_1 - 3a^3)\cos\tau - \frac{a^3}{12}\cos 3\tau. \quad (3.18)$$

Avoiding secular term, it gives

$$\omega_1 = \frac{a^2}{4}, \quad (3.19)$$

and the solution of Eq. (3.18) is

$$x_1 = \frac{a^3}{96}(-\cos\tau + \cos 3\tau). \quad (3.20)$$

By substituting the values of x_0 and x_1 from Eq. (3.17) and Eq. (3.20) into Eq. (3.16) and simplifying, it yields:

$$x_2'' + x_2 = \frac{1}{1920}\left((1920a\omega_2 - 155a^5)\cos\tau - 40a^5\cos 3\tau - 21a^5\cos 5\tau\right). \quad (3.21)$$

No secular term condition provides

$$\omega_2 = \frac{31a^4}{384}, \quad (3.22)$$

and the solution of Eq. (3.21) is obtained as

$$x_2 = \frac{a^5}{15360}(-47\cos\tau + 40\cos 3\tau + 7\cos 5\tau). \quad (3.23)$$

In the similar way, the obtained values are

$$\omega_3 = \frac{631a^6}{23040}, \quad (3.24)$$

$$x_3 = \frac{a^7}{10321920}(-9363\cos\tau + 6818\cos 3\tau + 2240\cos 5\tau + 305\cos 7\tau) \quad (3.25)$$

$$\omega_4 = \frac{393679a^8}{41287680}, \quad (3.26)$$

$$x_4 = \frac{a^9}{2972712960}(-827770\cos\tau + 519805\cos 3\tau + 239778\cos 5\tau + 61320\cos 7\tau + 6867\cos 9\tau), \quad (3.27)$$

$$\omega_5 = \frac{25131079a^{10}}{7431782400}, \quad (3.28)$$

$$x_5 = \frac{a^{11}}{5231974809600}(-463224472\cos\tau + 253530827\cos 3\tau + 145172720\cos 5\tau + 52356150\cos 7\tau + 11110176\cos 9\tau + 1054599\cos 11\tau) \quad (3.29)$$

$$\omega_6 = \frac{9547855309a^{12}}{7847962214400}, \quad (3.30)$$

$$x_6 = \frac{a^{13}}{6529504562380800}(-189316054776\cos\tau + 91361452910\cos 3\tau + 60642330749\cos 5\tau + 27407741504\cos 7\tau + 8280153882\cos 9\tau + 1500899400\cos 11\tau + 123476331\cos 13\tau) \quad (3.31)$$

$$\omega_7 = \frac{1264917158509a^{14}}{2856658246041600}, \quad (3.32)$$

$$x_7 = \frac{a^{15}}{21939135329599488000}(-213325644689686\cos\tau + 91801757752051\cos 3\tau + 67679167268895\cos 5\tau + 35921163222945\cos 7\tau + 13743810544464\cos 9\tau + 3573881897586\cos 11\tau + 564887294400\cos 13\tau + 40976709345\cos 15\tau) \quad (3.33)$$

$$\omega_8 = \frac{14278423131625999a^{16}}{87756541318397952000}. \quad (3.34)$$

Using all these values, the frequency Eq. (2.4) can be expressed as follows

$$\omega^2 = 1 + \frac{a^2}{4}\varepsilon + \frac{31a^4}{384}\varepsilon^2 + \frac{631a^6}{23040}\varepsilon^3 + \frac{393679a^8}{41287680}\varepsilon^4 + \frac{25131079a^{10}}{7431782400}\varepsilon^5 + \frac{9547855309a^{12}}{7847962214400}\varepsilon^6 + \frac{1264917158509a^{14}}{2856658246041600}\varepsilon^7 + \frac{14278423131625999a^{16}}{87756541318397952000}\varepsilon^8 + \cdots. \quad (3.35)$$

4. RESULTS AND DISCUSSION

The effectiveness of the proposed analytical approach is assessed through a detailed comparison of the obtained frequency and displacement responses with numerical results and existing analytical methods, including the harmonic balance method (HBM), frequency–amplitude formulation (FAF), and parameter expansion method (PEM). The numerical solutions are computed using the fourth-order Runge–Kutta method and are treated as benchmark results.

Table 4.1 compares the approximate frequencies obtained via the Lindstedt–Poincare method with the corresponding numerical results and previously reported values for various initial amplitudes. The accuracy of the analytical approximations is evaluated using the percentage error, denoted by Er(%). This error is computed as $\left|\frac{\omega_e-\omega_i}{\omega_e} \times 100\right|$, where ω_e is the exact frequency and ω_i is the i^th^ $(i = 0,1,2,\cdots)$ approximate frequency obtained by the present method.

It is observed that the present method yields highly accurate results over a wide range of amplitudes. For small amplitudes, the agreement between all methods is nearly exact, with negligible absolute percentage error. As the amplitude increases, the deviation of alternative methods such as HBM, FAF, and PEM becomes more pronounced. For instance, at $b = 3.0$, the error associated with these methods increases significantly (exceeding 13%), whereas the Lindstedt–Poincaré method maintains a much lower error (approximately 0.22%). This trend becomes even more evident at larger amplitudes; at $b = 7.0$, the error of the compared methods grows dramatically (over 300%), while the present method still provides reasonable accuracy with an error of about 3.04%. It is noted that the solution obtained by HBM, FAF, and PEM does not converge at $A \geq 7.32108$.

The graphical comparisons further confirm the superior performance of the present approach. Figures 4.1 and 4.2 illustrate the time-history responses for moderate amplitudes ($b = 1.0$ and $b = 1.5$). In these cases, the solutions obtained via the Lindstedt–Poincare method are almost indistinguishable from the numerical solutions, while the HBM, FAF, and PEM results exhibit slight deviations, particularly in phase. At larger amplitudes, as illustrated in Figure 4.3 for $b = 5.0$, the proposed method remains in close agreement with the numerical solution, demonstrating consistently high accuracy.

A more comprehensive evaluation is provided in Figure 4.4, where the absolute percentage errors of different methods are plotted against the amplitude. The results clearly indicate that the error associated with HBM, FAF, and PEM increases rapidly with amplitude, reflecting their limitations in handling strong nonlinearities. In contrast, the Lindstedt–Poincare method demonstrates a much slower growth in error, confirming its robustness and stability over a broader range of oscillation amplitudes.

Table 4.1: Comparison of the obtained results with the corresponding exact frequencies and previously reported results for the oscillator $\ddot{u} + (1 + \dot{u}^2)u = 0$, where Er% denotes the absolute percentage error.

b	ω_{Exact}	ω_{LP} / Er%	$\omega_{HBM}, \omega_{FAF}, \omega_{PEM}$ [13, 21, 22] / Er%
0.01	1.00001	1.00001	1.00001
		0.00	0.00
0.1	1.00125	1.00124	1.00125
		0.00	0.00
0.5	1.02848	1.02848	1.02912
		0.00	0.06
1.0	1.09230	1.09230	1.09982
		0.00	0.07
1.5	1.16367	1.16364	1.19070
		0.00	2.32
2.0	1.23149	1.23120	1.29354
		0.02	5.04
2.5	1.29320	1.29206	1.40754
		0.09	8.84
3.0	1.34887	1.34591	1.53510
		0.22	13.81
5.0	1.52501	1.50416	2.32197
		1.37	52.25
7.0	1.65291	1.60267	6.74288
		3.04	307.94
8.0	1.70546	1.63858	-----
		3.92	-----

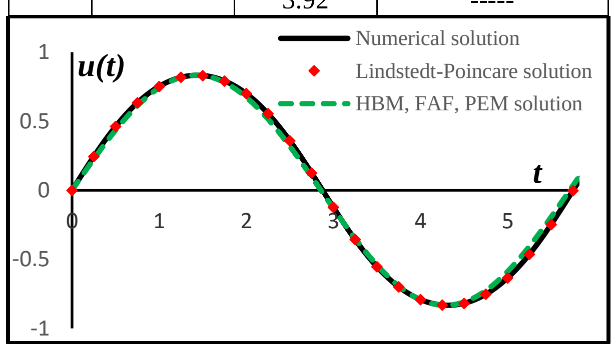

Figure 4.1: Comparison of the solution obtained by numerical methods, the Lindstedt–Poincaré method,

and other existing approaches [13, 21, 22] for the oscillator $\ddot{u} + (1 + \dot{u}^2)u = 0$ and $b = 1.0$.

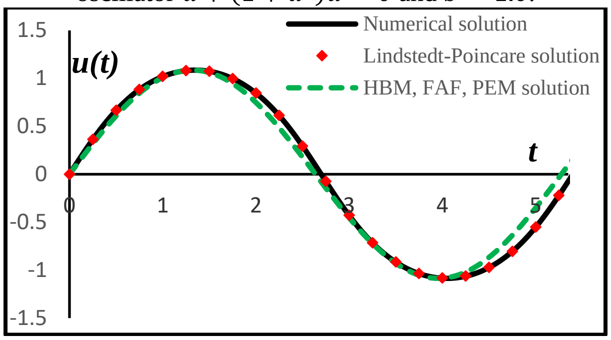

Figure 4.2: Comparison of the solution obtained by numerical methods, the Lindstedt–Poincaré method, and other existing approaches [13, 21, 22] for the oscillator $\ddot{u} + (1 + \dot{u}^2)u = 0$ and $b = 1.5$.

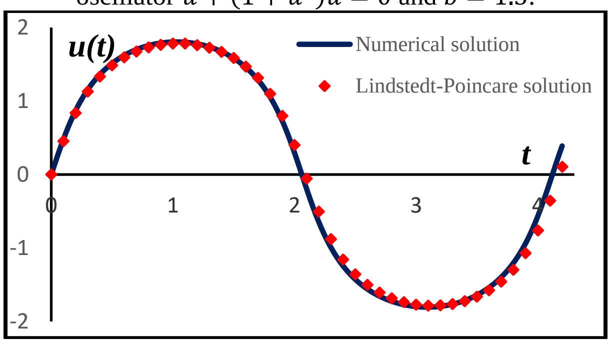

Figure 4.3: Comparison of the solution obtained by numerical methods and the Lindstedt–Poincaré method for the oscillator $\ddot{u} + (1 + \dot{u}^2)u = 0$ and $b = 5.0$.

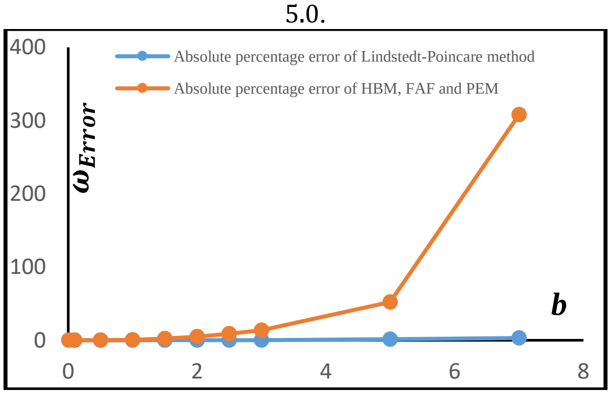

Figure 4.4: Comparison of the absolute percentage errors associated with the Lindstedt–Poincaré method and other existing approaches [13, 21, 22] for various values of b.

5. Conclusion

This paper has provided an analytical framework of studying a highly nonlinear oscillatory system by rewriting the original governing equation into a corresponding pendulum like equation. This work simplifies the mathematical treatment of the system and still retains the basic dynamical properties of the system which allows the application of classical analytical methods. Lindstedt-Poincare method successfully has been used to obtain approximate periodic solutions and the corresponding frequency-amplitude relationships. The approach removes the secular terms, and provides uniformly sound solutions over long durations of time. This characteristic renders it especially useful in the analysis of nonlinear oscillators with the amplitude-dependent behavior.

The validity and strength of the proposed method have been rigorously tested by comparing the results with numerical solutions to systems found using the fourth-order Runge-Kutta method, as well as with the results of other established analytic techniques, such as the harmonic balance technique, frequency-amplitude technique, and parameter expansion technique. The findings indicate that there is an excellent agreement of small and high amplitudes, and that the current method is significantly more accurate than the compared methods at higher amplitudes. Specifically, even in the highly nonlinear regime, the error growth is relatively small, which evidently demonstrates the reliability and stability of the approach.

In general, the current analysis proves that the combination of an appropriate transformation and the Lindstedt-Poincare method is a potent and effective tool to investigate nonlinear oscillatory systems. The methodology that has been developed in this work can be applied to a wider range of nonlinear problems that can be used to provide valuable insights into the dynamic behavior and frequency characteristics of these types of nonlinear problems.

Conflict of Interest

No conflict-of-interest present among the authors.

Data Availability

No datasets were generated or analyzed during the current study.

Funding Statement

The authors declare that no funds, grants, or other support were received during the preparation of this manuscript.

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