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Approximate solution of a nonlinear oscillator of mass attached to a stretched elastic wire

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ABSTRACT

This paper presents the application of the harmonic balance method to solve a nonlinear oscillator governing the oscillations of a mass attached to a stretched elastic wire. The oscillator, characterized by a restoring force parameter λ $(0 \le \lambda \le 1)$, serves as a model for confined systems where relativistic quantum effects are significant, including nuclear and quark models of matter and particle momentum. The method is also applicable to the study of massive photon propagation in space-time and quantum shell correction estimation. The proposed procedure is effective across the entire range of the parameter λ, showing excellent agreement with numerical simulations for both approximate frequencies and periodic solutions. Additionally, this approach is simplified by its avoidance of elliptic integrals, enhancing its utility and accessibility for researchers in the field.

1. Introduction

Nonlinear vibrations of many oscillatory systems are generally modelled by nonlinear differential equations. It is difficult to solve most of the nonlinear differential equations exactly and sometimes it is difficult to get analytic approximations than a numerical one. Many researchers give their time and significant attention to find approximate solutions of nonlinear equations by means of numerous analytical techniques. Earlier, several authors used analytical techniques named perturbation methods [1-6] in which the nonlinear response is very small. To overcome this specific limitation, some asymptotic expansion methods [7-14] have been developed by several authors. The homotopy perturbation method [15-19] is the powerful analytical technique for solving strong nonlinear problems. Also, different forms of iterative methods [20-26] have been used for solving strong nonlinear problems. The harmonic balance method [27-35] is another powerful technique for handling the nonlinear problems. But for the present oscillator, there arise much complexity in the homotopy perturbation method [19] and harmonic balance method [35] due to elliptic integrals [36]. The evaluation of elliptic integral is very difficult. However, in the present article this limitation is removed. In the present article, a new approach of harmonic balance method is applied to solve a nonlinear oscillator modelled from a mass attached to a stretched elastic wire. This oscillator contains an irrational restoring force [27, 37-38]. The present approach is very simple and yields a rapid convergence of the approximate frequencies and periodic solutions.

2. Solution procedure

Let us consider the nonlinear oscillator govern by the motion of a mass attached to a stretched elastic wire is [27]

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$\ddot{x} + x - \frac{\lambda x}{\sqrt{1+x^2}} = 0,\ 0 \le \lambda \le 1,$ (1)

where the initial conditions are

$x(0) = a$ and $\dot{x}(0) = 0.$ (2)

Let us consider the approximate solution of Eq. (1) of the truncated form of [35] as

$x(t) = a\big((1-u)\cos\varphi + u\cos 3\varphi\big),$ (3)

where $\varphi = \omega t$ in which ω is the angular frequency of the oscillator.

M.A. Razzak [35] used second order approximation Eq. (3) but, the equation for determining angular frequencies contains elliptic integrals that increase the difficulties of entire procedure. One of the aims of this paper is to reduce the algebraic complexity and to avoid elliptic integral. For this reason, we rewrite Eq. (1) of the form

$(1+x^2)(x'' + \tilde{x})^2 - \lambda^2\tilde{x}^2 = 0,\ 0 \le \lambda \le 1,$ (4)

with the same initial conditions in Eq. (2), where $\tilde{x} = x/\,a\cos\varphi$ and $x'' = \ddot{x}/\,a\cos\varphi$.

For simplicity letting

$\omega = \sqrt{1-\sqrt{\nu}}.$ (5)

Substituting Eq. (3) and (5) into Eq. (4) we obtained

$(1+x^2)(x'' + \tilde{x})^2 - \lambda^2\tilde{x}^2 = c_0 + c_2\cos 2\varphi + \cdots,$ (6)

where

$c_0 = (-\lambda^2 + \nu + \frac{a^2\nu}{2}) + (4\lambda^2 + 16\sqrt{\nu} - 20\nu - 2a^2\nu)u + (192 + 32a^2 - 6\lambda^2 - 448\sqrt{\nu} - 88a^2\sqrt{\nu} + 262\nu + 62a^2\nu)u^2,$ (7)

$c_2 = \frac{a^2\nu}{2} + (-4\lambda^2 - 32\sqrt{\nu} - 8a^2\sqrt{\nu} + 36\nu + 8a^2\nu)u + (-256 + 8\lambda^2 + 608\sqrt{\nu} + 24a^2\sqrt{\nu} - 360\nu - 27a^2\nu)u^2.$ (8)

Substituting Eq. (6) into Eq. (4) then equating the constant term and coefficient of $\cos 2\varphi$ to zero we obtained

$(-\lambda^2 + \nu + \frac{a^2\nu}{2}) + (4\lambda^2 + 16\sqrt{\nu} - 20\nu - 2a^2\nu)u + (192 + 32a^2 - 6\lambda^2 - 448\sqrt{\nu} - 88a^2\sqrt{\nu} + 262\nu + 62a^2\nu)u^2 = 0,$ (9)

$\frac{a^2\nu}{2} + (-4\lambda^2 - 32\sqrt{\nu} - 8a^2\sqrt{\nu} + 36\nu + 8a^2\nu)u + (-256 + 8\lambda^2 + 608\sqrt{\nu} + 24a^2\sqrt{\nu} - 360\nu - 27a^2\nu)u^2 = 0.$ (10)

If $u = 0$, then from Eq. (9) the zeroth order approximation of ν is obtained as

$\nu = \nu_0 = \frac{2\lambda^2}{2+a^2}.$ (11)

The Eq. (10) is a quadratic equation of u then, it is easily solved with the values of ν from Eq. (11). The first order approximation of ν is obtained from Eq. (9) as

$\nu = \nu_1 = \frac{2(\lambda^2 - \Delta)}{2+a^2},$ (12)

Where

$\Delta = (4\lambda^2 + 16\sqrt{\nu} - 20\nu - 2a^2\nu)u + (192 + 32a^2 - 6\lambda^2 - 448\sqrt{\nu} - 88a^2\sqrt{\nu} + 262\nu + 62a^2\nu)u^2$ (13)

For better approximation of u Eq. (10) can be used again with the first order approximation of ν obtained from Eq. (12).

3. Results and Discussion

In this paper, an analytical technique named harmonic balance method with a new simple approach is presented to investigate a strongly nonlinear oscillator govern by a mass attached to a stretched elastic wire. Recently, Belendez et. al. [19] has studied this nonlinear oscillator by homotopy perturbation method and M.A Razzak [35] has studied the same oscillator by harmonic balance method in a different approach. These methods contain much algebraic complexity due to the use of elliptic integrals. In the present study more emphasis is placed on this issue. Besides for checking the accuracy of the present approach, the approximate frequencies are calculated by a simple approach of Eq. (1) for several amplitudes of oscillation (with different values of the parameter λ) and are compared with the numerical results together with other existing results. All the results are shown in Tables 1–3. The tables show that the approximate frequencies of the present procedure provide good accuracy and nicely match together with the numerical results and other existing results. Also, the solutions for displacement of the present study are compared with numerical solution (fourth-order Runge–Kutta method) for various conditions which are presented in Figures 1-4. Seeing all Figures 1-4, it is clearly observed that the present solutions show a good coincidence with the corresponding numerical solutions for small and large values of amplitudes. All calculations were performed using Mathematica software.

Table 1 Comparison of the approximate frequencies (ω for $\nu = \nu_1$) obtained by the present method of (Eq. (1)) with the exact frequency ω_e and other existing frequencies for $\lambda = 0.5$.

a	ω_e	Belendez [19]	Razzak [35]	Present study
0.1	0.708423	0.708423	0.708423	0.708423
0.2	0.712259	0.712262	0.712259	0.712259
0.4	0.726126	0.726162	0.726125	0.726122
0.6	0.745140	0.745269	0.745143	0.745117
0.8	0.765907	0.766147	0.765896	0.765806
1	0.786171	0.786524	0.786150	0.785945
2	0.860447	0.860963	0.860386	0.859485
5	0.937317	0.937529	0.937281	0.936705
10	0.968102	0.968168	0.968091	0.968018
100	0.996812	0.996813	0.996812	0.996831

Table 2 Comparison of the approximate frequencies (ω for $\nu = \nu_1$) obtained by the present method of (Eq. (1)) with the exact frequency ω_e and other existing frequencies for $\lambda = 0.75$.

a	ω_e	Belendez [19]	Razzak [35]	Present study
0.1	0.502786	0.502787	0.502786	0.502786
0.2	0.510841	0.510859	0.510840	0.510840
0.4	0.539214	0.539413	0.539212	0.539207
0.6	0.576587	0.577182	0.576578	0.576530
0.8	0.615781	0.616824	0.615753	0.615578
1	0.652771	0.654164	0.652715	0.652311
2	0.780662	0.782231	0.780510	0.778744
5	0.904141	0.904677	0.904059	0.902921
10	0.951696	0.951853	0.951669	0.951483
100	0.995214	0.995215	0.995213	0.995242

Table 3 Comparison of the approximate frequencies (ω for $\nu = \nu_1$) obtained by the present method of (Eq. (1)) with the exact frequency ω_e and other existing frequencies for $\lambda = 0.95$.

a	ω_e	Belendez [19]	Razzak [35]	Present study
0.1	0.231367	0.231388	0.231367	0.231367
0.2	0.252549	0.252792	0.252553	0.252559
0.4	0.317642	0.329204	0.317677	0.317752
0.6	0.391035	0.394094	0.391092	0.391177
0.8	0.459947	0.463965	0.459975	0.459851
1	0.520335	0.524765	0.520302	0.519758
2	0.709629	0.713013	0.709385	0.706557
5	0.876561	0.877510	0.876426	0.874652
10	0.938333	0.938597	0.938290	0.937967
100	0.993933	0.993935	0.993932	0.993968

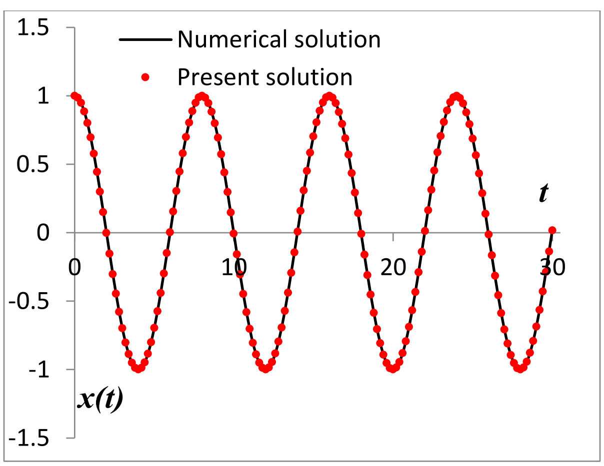

Figure 1. Comparison of time versus displacement curve for Eq. (1) obtained by present method and numerical method for $x(0) = 1, \dot{x}(0) = 0, \lambda = 0.5$.

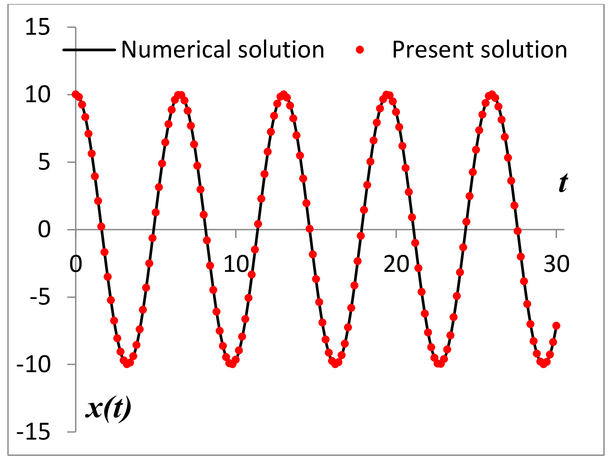

Figure 2. Comparison of time versus displacement curve for Eq. (1) obtained by present method and numerical method for $x(0) = 10, \dot{x}(0) = 0, \lambda = 0.5$.

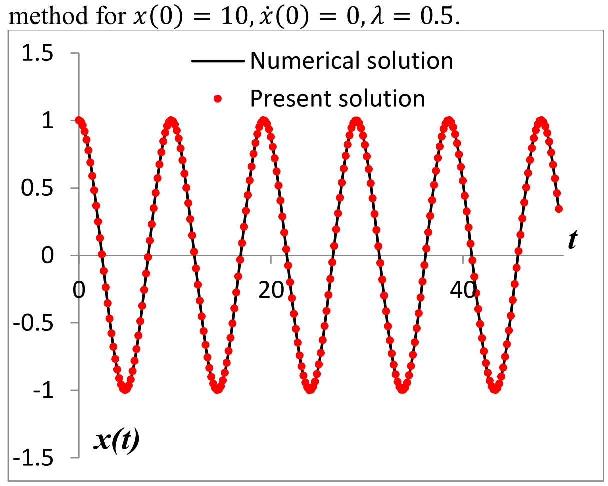

Figure 3. Comparison of time versus displacement curve for Eq. (1) obtained by present method and numerical method for $x(0) = 1, \dot{x}(0) = 0, \lambda = 0.75$.

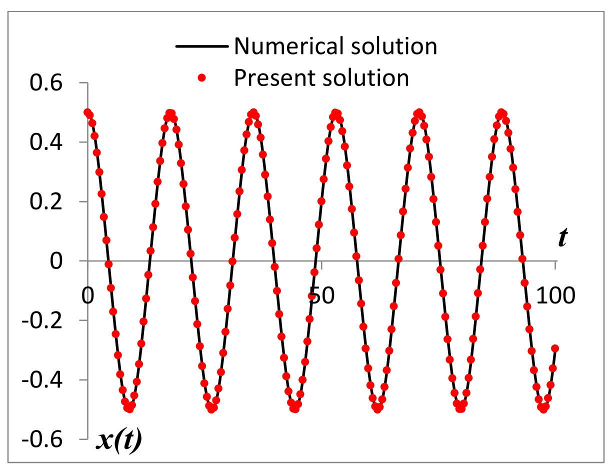

Figure 4. Comparison of time versus displacement curve for Eq. (1) obtained by present method and numerical method for $x(0) = 0.5, \dot{x}(0) = 0, \lambda = 0.95$.

4. Conclusion

A new analytical approach of harmonic balance method is presented for determining the approximate frequencies as well as the approximate periodic solutions of strongly nonlinear oscillator govern by a mass attached to a stretched elastic wire. The main advantage of this paper is its simplicity. Also, the results obtained by present method are nicely coincidence with the results corresponding to numerical one for both small and large values of amplitudes. Though the present approach is exemplified for a strongly nonlinear oscillator of a mass attached to a stretched elastic wire, it is also applicable for parallel strongly nonlinear oscillators for example Duffing-harmonic oscillator and relativistic oscillator.

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Declaration of interest statement:

The authors declare that they have no conflict of interests.

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