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Analytical Technique for Solving Van der Pol Oscillator

Md. Zahangir Alam[1*], Ismot Ara Yeasmin[1], Md. Nazmul Sharif[1]

[1]*Department of Mathematics, Rajshahi University of Engineering & Technology (RUET), Rajshahi-6204*

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ABSTRACT

An analytical technique like Differential Transform Method (DTM) is presented to find out the solution of Van der Pol oscillator. In the proposed method, the algebraic relations for determining the unknown coefficients are linear equations and independent of trigonometric functions. To justify the effectiveness and suitability of the method, the obtained solutions are matched to corresponding numerical solutions for several damping conditions. The method proposed herein is straightforward and proficient technique for determining approximate solutions of such kind of nonlinear oscillators.

1. Introduction

Numerous physical and engineering systems are moulded by nonlinear differential equations. Most of these equations arise in various natural and man-made systems and have no exact solutions. Several nonlinear oscillators were analysed by numerous authors using various approaches [1-13], such as the perturbation technique, the homotopy perturbation method, the variational iteration method (VIM), the harmonic balance method (HBM), the energy balance method etc. The perturbation methods [1-7] were originally developed for solving weakly nonlinear problems. In contrast, the homotopy perturbation method [8-10], the variational iteration method (VIM) [11], the harmonic balance method (HBM) [12], the energy balance method [13] etc. were widely used method for handling strongly nonlinear oscillators. It is noticeable that most of the cases the mathematical formulation of these methods contains trigonometric functions.

A semi-analytical procedure known as differential transform method (DTM) [14-23] was employed to find periodic solutions for certain strongly nonlinear oscillators. But the mathematical manipulation is very laborious task. Recently, a new analytical technique like DTM [24-25] has been developed to overcome this limitation.

In this article, an analytical technique like DTM [24-25] with its modification is presented for solving Van der Pol oscillators. The mathematical formulations are extremely straightforward and free from trigonometric functions. The solutions obtained by present method are well agreed with the solutions obtained by numerical one.

* Corresponding authors: Department of Mathematics, Rajshahi University of Engineering & Technology (RUET), Rajshahi-6204
E-mail addresses: zahangir@math.ruet.ac.bd (Md. Zahangir Alam)

2. The Method

Let's examine the Van der Pol oscillator

$$\ddot{x} - \varepsilon(1 - x^2)\dot{x} + x = 0, x(0) = 0, \dot{x}(0) = b, \quad (1)$$

where dots above the variables indicate differentiation w. r. to t.

The polynomial-type solution of Eq. (1) was identified in [24] as

$$x(\tau) = B_1(b, \omega_0, \varepsilon, \omega)\varphi(\tau) + B_2(b, \omega_0, \varepsilon, \omega)\varphi^2(\tau) + B_3(b, \omega_0, \varepsilon, \omega)\varphi^3(\tau) + \cdots, \quad (2)$$

where $\varphi(\tau) = \tau(\pi - \tau)$ and $\tau = \omega t$ was considered as a new variable in place of t.

In this article, we propose the estimated solution of Eq. (1) as

$$x(t) = B_1(b, \varepsilon)\varphi(t) + B_2(b, \varepsilon)\varphi^2(t) + B_3(b, \varepsilon)\varphi^3(t) + \cdots, \quad (3)$$

where $\varphi(t) = t(S - t), B_i, i = 1, 2, 3, \cdots$ are unknowns. In the article [24], $S = \pi$ is chosen but it is not always true. For this oscillator $2S$ is considered as period of oscillation. The approximate solution Eq. (3) is valid for the semi period, $0 \le t \le S$. And it is valid for the next semi period $S \le t \le 2S$, if the values of φ and b be switched by $\varphi(t) = (t - S)(2S - t)$ and $-b$ respectively. In the case of Van der Pol oscillator, the solution about $t = \frac{S}{2}$ is not symmetric. In this situation, each semi period can be divided into two sub-intervals. The first valid sub-interval is chosen as $0 < t < \frac{S_1}{2}$ where $\varphi(t) = t(S_1 - t)$. The next sub-interval is chosen, $\frac{S_1}{2} < t < S$ where $\varphi(t) = t(S_2 - t), \frac{S_2}{2} = S - \frac{S_1}{2}$. For this sub-interval, the valid solution is obtained from the new form of Eq. (1) [24].

The maximum displacement $x(t)$ occurs when $\varphi(t)$ becomes maximum. The supreme value of $\varphi(t)$ at the first sub-intervals is $\frac{{S_1}^2}{4}$ when $t = \frac{S_1}{2}$. And the supreme value of $\varphi(t)$ at the next sub-intervals is $\frac{{S_2}^2}{4}$ when $t = \frac{S_2}{2}$ in the transformed form of Eq. (1) [$t = \frac{S_2}{2}$ in the transform form is the same point $t = \frac{S_1}{2}$ in the original form]. Let a be the amplitude of the oscillation then following [24], it can be written as

$$a = \frac{B_1(b,\ \varepsilon){S_1}^2}{4} + \frac{B_2(b,\ \varepsilon){S_1}^4}{16} + \frac{B_3(b,\ \varepsilon){S_1}^6}{64} + \cdots, \quad (4)$$

and the similar equation is obtained for the next sub-intervals.

To determine B_1, differentiating Eq. (3) w. r. to t, yields

$$x'(t) = \left(\sqrt{{S_1}^2 - 4\varphi(t)}\right)(B_1 + 2B_2\varphi(t) + 3B_3\varphi(t)^2 + \cdots). \quad (5)$$

Using the initial condition $x'(0) = b$, (since $\varphi(t) \to 0$ as $t \to 0$) the Eq. (5) gives

$$B_1 = \frac{b}{S_1}. \quad (6)$$

By substituting Eq. (3) into Eq. (1) and matching the similar coefficients like φ^0, φ^1, $\varphi^2, \cdots$ a sequence of linear equations containing $B_i, i = 2, 3, \cdots$ are obtained as follows

$$\mathrm{B}_2 = \frac{b}{{S_1}^3} + \frac{b\varepsilon}{2{S_1}^2}, \quad (7)$$

$$\mathrm{B}_3 = \frac{2b}{{S_1}^5} - \frac{b}{6{S_1}^3} + \frac{b\varepsilon}{{S_1}^4} + \frac{b\varepsilon^2}{6{S_1}^3}, \quad (8)$$

$$\mathrm{B}_4 = \frac{5b}{{S_1}^7} - \frac{b}{2{S_1}^5} + \frac{5b\varepsilon}{2{S_1}^6} - \frac{b\varepsilon}{12{S_1}^4} - \frac{b^3\varepsilon}{12{S_1}^4} + \frac{b\varepsilon^2}{2{S_1}^5} + \frac{b\varepsilon^3}{24{S_1}^4}, \quad (9)$$

The Eq. (1) must satisfy the relation bellow

$$b^2 = a^2 + D \text{ or } b = \sqrt{a^2 + D} \quad (10)$$

where $D = -\int_0^{\frac{{S_1}^2}{4}} 2\varepsilon(1 - x^2)\dot{x}^2\, d\varphi$.

3. Results and Discussion

An analytical technique like Differential Transform method is presented for solving Van der Pol oscillator. In the present method, all equations for the unknown coefficients are linear. Consequently, the mathematical operations involved in this method are significantly simpler compared to other existing methods. Moreover, the results produced by this method closely match the corresponding numerical results found by the fourth-order Runge-Kutta method.

It is mentioned that, the amplitude a for the Van der Pol oscillator is approximately equal to 2. For steady solution, one can easily calculate b and S_1 using Eq. (4) and (10) by proper choice of b and $a = 2$. Letting, $b = 2, a = 2$ and $\varepsilon = 0.15$ Eq. (4) gives $S_1 \cong 2.963$ and then Eq. (10) provides $b \cong 2.005$. Similarly for the next sub-interval $S_2 \cong 3.3295$ is found. Thus, the approximate value of one semi period of the oscillation by the present method is obtained as $S = \frac{S_1}{2} + \frac{S_2}{2} \cong 3.146$. In this situation, the numerical methods provide the semi period which is approximately 3.146. It is observed that the present semi period is almost equal to numerical semi period. It is noted that both values are different from π. For $\varepsilon = 0.25$ Eq. (4) gives $S_1 \cong 2.83$ and then Eq. (10) provides $b \cong 2.015$ and for the next sub-interval $S_2 \cong 3.478$ is found. Similarly, for $b = 2$, $a = 2$ and $\varepsilon = 0.50$ the approximate values are found as $S_1 \cong 2.60$, $S_2 \cong 3.84$ and $b = 2.0661$. It is clearly noted that the present method provides suitable results for $0 \le \varepsilon < \frac{1}{2}$.

The Figure 1-3 show the comparison of time versus displacement curve for Eq. (1). All the figures are plotted by the results obtained by present method alongside those obtained through numerical methods. It is observed that, the present method shows accurate results and nicely matches with corresponding numerical one.

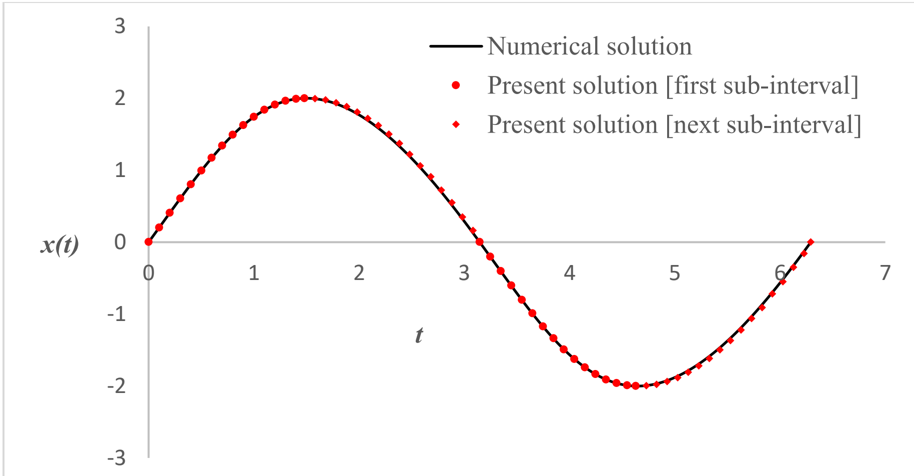

Figure 1. Comparison of time versus displacement curve for Eq. (1) obtained by present method and numerical method for $x(0) = 0, \dot{x}(0) = 2.005, \varepsilon = 0.15$.

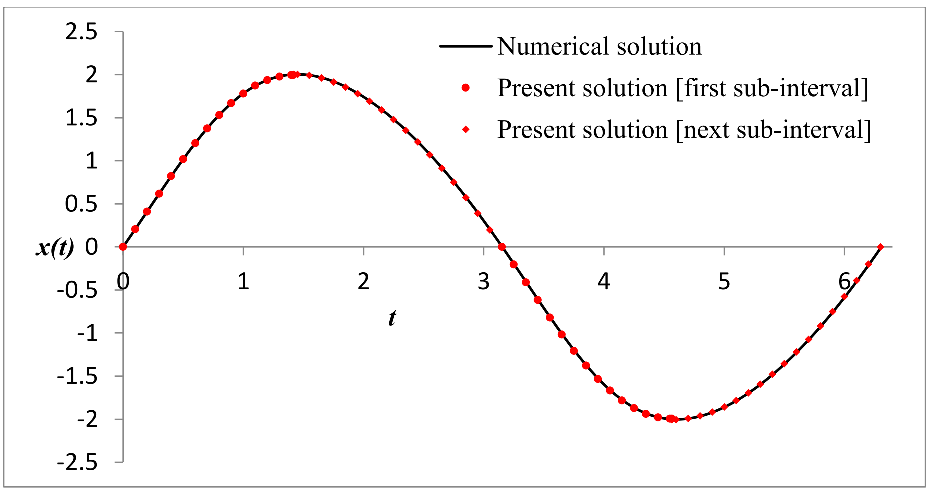

Figure 2. Comparison of time versus displacement curve for Eq. (1) obtained by present method and numerical method for $x(0) = 0, \dot{x}(0) = 2.015, \varepsilon = 0.25$.

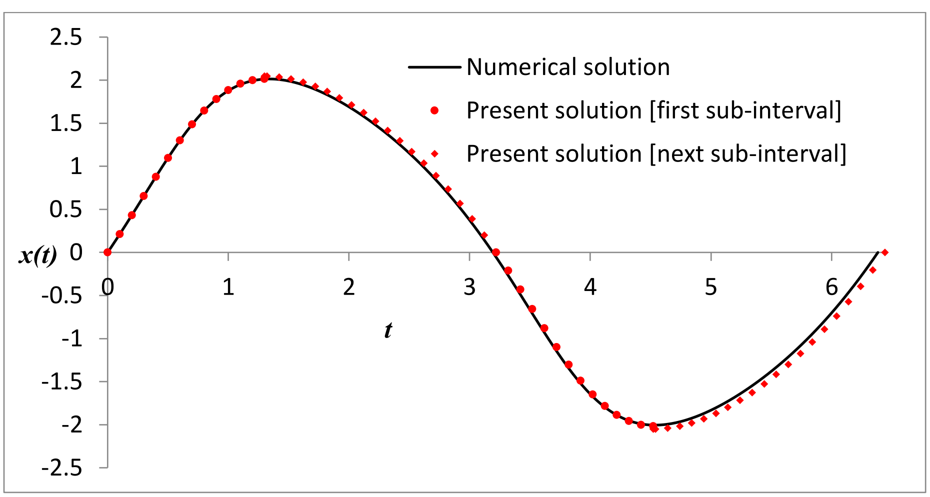

Figure 3. Comparison of time versus displacement curve for Eq. (1) obtained by present method and numerical method for $x(0) = 0, \dot{x}(0) = 2.0661, \varepsilon = 0.5$.

4. Conclusion

A methodological approach akin to the Differential Transform method is introduced to tackle the complexity of the extremely nonlinear Van der Pol oscillator. This technique offers simplicity in determining unknown coefficients due to the linear nature of the mathematical relations so formed. Moreover, it yields satisfactory outcomes across various conditions within the range of $0 \leq \varepsilon < \frac{1}{2}$, demonstrating commendable consistency with numerical solutions. The suggested method stands out as an effective tool for resolving nonlinear problems of this nature.

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